

A q -ANALOGUE OF AN IDENTITY OF N.WALLACH

G. LUSZTIG

1. Let n be an integer ≥ 2 . For $i \in [1, n-1]$ let s_i be the transposition $(i, i+1)$ in the group S_n of permutations of $[1, n]$. (Given two integers a, b we denote by $[a, b]$ the set of all integers c such that $a \leq c \leq b$.) Consider the following element of $\mathbf{C}[S_n]$ (the group algebra of S_n):

$$\mathbf{t} = s_1 s_2 \dots s_{n-1} + s_2 s_3 \dots s_{n-1} + \dots + s_{n-2} s_{n-1} + s_{n-1} + 1.$$

(The sum of an n -cycle, an $n-1$ -cycle, ..., a 2-cycle and the identity.) Wallach [W] proved the remarkable identity

$$(a) \quad \mathbf{t} \prod_{k \in [1, n], k \neq n-1} (\mathbf{t} - k) = 0$$

in $\mathbf{C}[S_n]$ and used it to establish a vanishing result for some Lie algebra cohomologies. In particular, left multiplication by \mathbf{t} in $\mathbf{C}[S_n]$ has eigenvalues in $\{0, 1, 2, \dots, n-2, n\}$. A closely related result appeared later in connection with a problem concerning shuffling of cards in Diaconis, Fill and Pitman [DFP] and also in Phatarfod [Ph].

Let \mathbf{q} be an indeterminate. Let H be the $\mathbf{Z}[\mathbf{q}]$ -algebra with generators T_1, T_2, \dots, T_{n-1} and relations $(T_i + 1)(T_i - \mathbf{q}) = 0$ for $i \in [1, n-1]$, $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for $i \in [1, n-2]$, $T_i T_j = T_j T_i$ for $i \neq j$ in $[1, n-1]$, a Hecke algebra of type A_{n-1} . Set

$$\tau = T_1 T_2 \dots T_{n-1} + T_2 T_3 \dots T_{n-1} + \dots + T_{n-2} T_{n-1} + T_{n-1} + 1 \in H.$$

Under the specialization $\mathbf{q} = 1$, τ becomes the element \mathbf{t} of $\mathbf{C}[S_n]$. Our main result is the following q -analogue of (a):

Supported in part by the National Science Foundation

Proposition 2. *The following equality in H holds:*

$$\tau \prod_{k \in [1, n], k \neq n-1} (\tau - 1 - \mathbf{q} - \mathbf{q}^2 - \dots - \mathbf{q}^{k-1}) = 0.$$

The proof will be given in Section 4. The proof of the Proposition is a generalization of the proof of 1(a) given in [GW]. However, there is a new difficulty due to the fact that the product of two standard basis elements of H is not a standard basis element (as for S_n) but a complicated linear combination of basis elements. To overcome this difficulty we will work in a model of H as a space of functions on a product of two flag manifolds over a finite field.

Let V be a vector space of dimension n over a finite field \mathbf{F}_q of cardinal q . Let \mathcal{F} be the set of complete flags

$$V_* = (V_0 \subset V_1 \subset V_2 \subset \dots \subset V_n)$$

in V where V_k is a subspace of V of dimension k for $k \in [0, n]$. Now $GL(V)$ acts on \mathcal{F} by

$$g : V_* \mapsto gV_* = (gV_0 \subset gV_1 \subset gV_2 \subset \dots \subset gV_n)$$

and on $\mathcal{F} \times \mathcal{F}$ by $g : (V_*, V'_*) \mapsto (gV_*, gV'_*)$. Let \mathcal{H} be the \mathbf{C} -vector space of all functions $f : \mathcal{F} \times \mathcal{F} \rightarrow \mathbf{C}$ that are constant on the orbits of $GL(V)$. This is an associative algebra with multiplication

$$f, f' \mapsto f * f', \quad (f * f')(W_*, V'_*) = \sum_{V'_* \in \mathcal{F}} f(W_*, V'_*) f'(V'_*, V_*).$$

Define $f_1 \in \mathcal{H}$ by

$$\begin{aligned} f_1(W_*, V'_*) &= 1 \text{ if there exists } g \in [1, n] \text{ (necessarily unique) with } W_r = V'_r \text{ for} \\ &r \in [1, g-1], V'_r \neq W_r \subset V'_{r+1} \text{ for } r \in [g, n-1]; \\ f_1(W_*, V'_*) &= 0, \text{ otherwise.} \end{aligned}$$

For $t \in [0, n]$ and any sequence $1 \leq i_1 < i_2 < \dots < i_{n-t} \leq n$ let $X_t^{i_1, i_2, \dots, i_{n-t}}$ be the set of all pairs $(V'_*, V_*) \in \mathcal{F} \times \mathcal{F}$ such that $V'_r \subset V_{i_r}, V'_r \not\subset V_{i_r-1}$ for $r \in [1, n-t]$. For $t \in [0, n]$ let $X_t = \cup X_t^{i_1, i_2, \dots, i_{n-t}} \subset \mathcal{F} \times \mathcal{F}$ where the union is taken over all sequences $1 \leq i_1 < i_2 < \dots < i_{n-t} \leq n$. Clearly, this union is disjoint and $X_0 \subset X_1 \subset X_2 \subset \dots \subset X_n = \mathcal{F} \times \mathcal{F}$. Also, X_0 is the diagonal in $\mathcal{F} \times \mathcal{F}$. Define $f_t \in \mathcal{H}$ by

$$f_t(V'_*, V_*) = 1 \text{ if } (V'_*, V_*) \in X_t, f_t(V'_*, V_*) = 0, \text{ otherwise.}$$

For $t = 1$ this agrees with the earlier definition of f_1 . Note that f_0 is the unit element of \mathcal{H} . The following result is a q -analogue of a result in [DFP].

Lemma 3. *For $t \in [1, n-1]$ we have $f_1 * f_t = (1 + q + q^2 + \dots + q^{t-1})f_t + q^t f_{t+1}$.*

Let $f = f_1 * f_t$. From the definitions we have $f = \sum_{g=1}^n \phi_g$ where $\phi_g \in \mathcal{H}$ is defined as follows: for $(W_*, V_*) \in \mathcal{F} \times \mathcal{F}$, $\phi_g(W_*, V_*)$ is the number of $V'_* \in \mathcal{F}$ such that

$V'_r = W_r$ for $r \in [1, g-1]$, $V'_r \neq W_r \subset V'_{r+1}$ for $r \in [g, n-1]$ and there exists $1 \leq i_1 < i_2 < \dots < i_{n-t} \leq n$ with $V'_r \subset V_{i_r}$, $V'_r \not\subset V_{i_{r-1}}$ for $r \in [1, n-t]$. Here V'_r is uniquely determined for $r \in [1, g-1]$ (we have $V'_r = W_r$) while for $r \in [g+1, n-1]$, V'_r is equal to $V'_g + W_{r-1}$ (this follows by induction from $V'_r = V'_{r-1} + W_{r-1}$ which holds since V'_{r-1}, W_{r-1} must be distinct hyperplanes of V'_r). Hence $\phi_g(W_*, V_*)$ is the cardinal of the set Y_g consisting of all g -dimensional subspaces V'_g of V such that

$$W_{g-1} \subset V'_g,$$

$V'_g + W_{r-1} \neq W_r$ for $r \in [g, n-1]$ (or equivalently $V'_g \not\subset W_{n-1}$), and there exists $1 \leq i_1 < i_2 < \dots < i_{n-t} \leq n$ (necessarily unique) with

$$W_r \subset V_{i_r}, W_r \not\subset V_{i_{r-1}} \text{ if } r \in [1, n-t] \cap [1, g-1],$$

$$V'_g \subset V_{i_g}, V'_g \not\subset V_{i_{g-1}} \text{ if } g \in [1, n-t],$$

$$V'_g + W_{r-1} \subset V_{i_r}, V'_g + W_{r-1} \not\subset V_{i_{r-1}} \text{ if } r \in [1, n-t] \cap [g+1, n-1].$$

Assume first that $g \in [1, n-t]$. If a $V'_g \in Y_g$ exists and if $1 \leq i_1 < i_2 < \dots < i_{n-t} \leq n$ is as above then, setting $j_r = i_r$ for $r \in [1, g-1]$ and $j_r = i_{r+1}$ for $r \in [g, n-t-1]$, we have $1 \leq j_1 < j_2 < \dots < j_{n-t-1} \leq n$ and

$$(a) \ W_r \subset V_{j_r}, W_r \not\subset V_{j_{r-1}} \text{ for } r \in [1, n-t-1].$$

(For $r \in [1, g-1]$ this is clear. Assume now that $r \in [g, n-t-1]$. Since $V'_g + W_r \subset V_{j_r}$, we have $W_r \subset V_{j_r}$. If $W_r \subset V_{j_{r-1}}$ then, since $V'_g \subset V_{i_g} \subset V_{j_{r-1}}$ and $j_r = i_{r+1}$, we would have $V'_g + W_r \subset V_{i_{r+1}-1}$, contradiction.) We see that $\phi_g(W_*, V_*) = 0$ if $(W_*, V_*) \notin X_{t+1}$. We now assume that $(W_*, V_*) \in X_{t+1}$. Let $1 \leq j_1 < j_2 < \dots < j_{n-t-1} \leq n$ be such that (a) holds. Then $\phi_g(W_*, V_*)$ is the number of g -dimensional subspaces V'_g of V such that

$$(b) \ W_{g-1} \subset V'_g \not\subset W_{n-1},$$

and

$$(c) \text{ if } g = 1 \leq n-t-1 \text{ then } V'_g \subset V_i, V'_g \not\subset V_{i-1} \text{ for some } i \text{ with } 1 \leq i < j_g;$$

$$(d) \text{ if } g \in [2, n-t-1] \text{ then } V'_g \subset V_i, V'_g \not\subset V_{i-1} \text{ for some } i \text{ with } j_{g-1} < i < j_g;$$

$$(e) \text{ if } g = n-t \geq 2 \text{ then } V'_g \subset V_i, V'_g \not\subset V_{i-1} \text{ for some } i \text{ with } j_{g-1} < i \leq n.$$

Now conditions (c),(d),(e) can be replaced by:

$$(c') \text{ if } g = 1 \leq n-t-1 \text{ then } V'_g \subset V_{j_{g-1}};$$

$$(d') \text{ if } g \in [2, n-t-1] \text{ then } V'_g \subset V_{j_{g-1}}, V'_g \not\subset V_{j_{g-1}-1};$$

$$(e') \text{ if } g = n-t \geq 2 \text{ then } V'_g \not\subset V_{j_{g-1}}.$$

Setting $L = V'_g/W_{g-1}$ we see that $\phi_g(W_*, V_*)$ is the number of lines L in V/W_{g-1} such that $L \not\subset W_{n-1}/W_{g-1}$ and

$$\text{if } g = 1 \leq n-t-1 \text{ then } L \subset V_{j_{g-1}}/W_{g-1};$$

$$\text{if } g \in [2, n-t-1] \text{ then } L \subset V_{j_{g-1}}/W_{g-1}, L \not\subset V_{j_{g-1}-1}/W_{g-1};$$

$$\text{if } g = n-t \geq 2 \text{ then } L \not\subset V_{j_{g-1}}/W_{g-1}.$$

Since W_{n-1}/W_{g-1} is a hyperplane in V/W_{g-1} , we see that $\phi_g(W_*, V_*)$ is given by:

$$(q^{j_g-g} - q^{j_g-g-1})/(q-1) = q^{j_g-g-1} \text{ if } g = 1 \leq n-t-1 \text{ and } V_{j_{g-1}} \not\subset W_{n-1},$$

$$0 \text{ if } g = 1 \leq n-t-1 \text{ and } V_{j_{g-1}} \subset W_{n-1},$$

$$(q^{j_g-g} - q^{j_g-g-1} - q^{j_{g-1}-g+1} + q^{j_{g-1}-g})/(q-1) = q^{j_g-g-1} - q^{j_{g-1}-g} \text{ if } g \in [2, n-t-1] \text{ and } V_{j_{g-1}} \not\subset W_{n-1},$$

$(q^{j_g-g} - q^{j_g-g-1})/(q-1) = q^{j_g-g-1}$ if $g \in [2, n-t-1]$ and $V_{j_g-1} \subset W_{n-1}, V_{j_g-1} \not\subset W_{n-1}$,
 0 if $g \in [2, n-t-1]$ and $V_{j_g-1} \subset W_{n-1}$,
 $(q^{n-g+1} - q^{n-g} - q^{j_{g-1}-g+1} + q^{j_{g-1}-g})/(q-1) = q^{n-g} - q^{j_{g-1}-g}$ if $g = n-t \geq 2$
 and $V_{j_{g-1}} \not\subset W_{n-1}$,
 q^{n-g} if $g = n-t \geq 2$ and $V_{j_{g-1}} \subset W_{n-1}$,
 q^{n-g} if $g = 1 = n-t$.

Now there is a unique $u \in [1, n]$ such that $V_{u-1} \subset W_{n-1}, V_u \not\subset W_{n-1}$. From (a) we see that $u \notin \{j_1, j_2, \dots, j_{n-t-1}\}$ (we use that $n-t-1 < n-1$). Using the formulas above, we can now compute $N = \sum_{g \in [1, n-t]} \phi_g(W_*, V_*)$.

If $u < j_1$ and $n-t \geq 2$ (so that $V_{j_1-1} \not\subset W_{n-1}$) we have

$$N = q^{j_1-2} + \sum_{g=2}^{n-t-1} (q^{j_g-g-1} - q^{j_{g-1}-g}) + (q^t - q^{j_{n-t-1}-1}) = q^t.$$

If $j_{h-1} < u < j_h$ for some $h \in [2, n-t-1]$ (so that $V_{j_{h-1}} \subset W_{n-1}, V_{j_h-1} \not\subset W_{n-1}$) we have

$$N = q^{j_h-h-1} + \sum_{g=h+1}^{n-t-1} (q^{j_g-g-1} - q^{j_{g-1}-g}) + (q^t - q^{j_{n-t-1}-1}) = q^t.$$

If $j_{n-t-1} < u$ and $n-t \geq 2$ (so that $V_{j_{n-t-1}} \subset W_{n-1}$) we have $N = q^t$.

If $n-t = 1$ we have $N = q^t$.

We see that in any case we have $N = q^t$.

Assume next that $g \in [n-t+1, n]$. If a $V'_g \in Y_g$ exists then there exists $1 \leq i_1 < i_2 < \dots < i_{n-t} \leq n$ such that

(f) $W_r \subset V_{i_r}, W_r \not\subset V_{i_r-1}$ if $r \in [1, n-t]$.

We see that $\phi_g(W_*, V_*) = 0$ if $(W_*, V_*) \notin X_t$. We now assume that $(W_*, V_*) \in X_t$ and that $1 \leq i_1 < i_2 < \dots < i_{n-t} \leq n$ is such that (f) holds. Then $\phi_g(W_*, V_*)$ is the number of g -dimensional subspaces V'_g of V such that

$$W_{g-1} \subset V'_g \not\subset W_{n-1}$$

that is, the number of lines L in V/W_{g-1} such that $L \not\subset W_{n-1}/W_{g-1}$. We see that $\phi_g(W_*, V_*) = q^{n-g}$. Hence $\sum_{g \in [n-t+1, n]} \phi_g(W_*, V_*) = 1 + q + q^2 + \dots + q^{t-1}$.

Summarizing, we see that for $(W_*, V_*) \in \mathcal{F} \times \mathcal{F}$, $f(W_*, V_*) = \sum_{g=1}^n \phi_g(W_*, V_*)$ is equal to

$$\begin{aligned}
 &1 + q + q^2 + \dots + q^t \text{ if } (W_*, V_*) \in X_t, \\
 &q^t \text{ if } (W_*, V_*) \in X_{t+1} - X_t, \\
 &0 \text{ if } (W_*, V_*) \notin X_{t+1}.
 \end{aligned}$$

The lemma follows immediately.

4. We show that

(a) $q^{1+2+\dots+(t-1)} f_t = f_1 * (f_1 - 1) * (f_1 - 1 - q) * \dots * (f_1 - 1 - q - q^2 - \dots - q^{t-2})$ for $t \in [1, n-1]$ by induction on t . For $t = 1$ this is clear. Assume that $t \in [2, n-1]$ and that (a) holds when t is replaced by $t-1$. Using Lemma 3 we have $q^{t-1} f_t = (f_1 - 1 - q - q^2 - \dots - q^{t-2}) * f_{t-1}$. Using this and the induction hypothesis we have

$$\begin{aligned}
 q^{1+2+\dots+(t-1)} f_t &= (f_1 - 1 - q - q^2 - \dots - q^{t-2}) * f_1 * (f_1 - 1) * \\
 &\quad * (f_1 - 1 - q) * \dots * (f_1 - 1 - q - q^2 - \dots - q^{t-3}).
 \end{aligned}$$

This proves (a).

Next we note that X_{n-1} is the set of all $(V'_*, V_*) \in \mathcal{F} \times \mathcal{F}$ such that for some $i \in [1, n]$ we have $V'_1 \subset V_i, V'_1 \not\subset V_{i-1}$. Thus, $X_{n-1} = \mathcal{F} \times \mathcal{F} = X_n$ so that $f_{n-1} = f_n$. Using this and Lemma 3 we see that $f_1 * f_{n-1} = (1 + q + q^2 + \cdots + q^{n-1})f_{n-1}$ that is $(f_1 - 1 - q - q^2 - \cdots - q^{n-1})f_{n-1} = 0$. Hence multiplying both sides of (a) (for $t = n - 1$) by $(f_1 - 1 - q - q^2 - \cdots - q^{n-1})$ we obtain

$$\begin{aligned} & f_1 * (f_1 - 1) * (f_1 - 1 - q) * \cdots * \\ & * (f_1 - 1 - q - q^2 - \cdots - q^{n-3}) * (f_1 - 1 - q - q^2 - \cdots - q^{n-1}) = 0. \end{aligned}$$

Thus an identity like that in Proposition 2 holds in \mathcal{H} instead of H (with f_1, q instead of τ, \mathbf{q}). It is known that the algebra \mathcal{H} may be identified with $\mathbf{C} \otimes_{\mathbf{Z}[\mathbf{q}]} H$ (where \mathbf{C} is regarded as a $\mathbf{Z}[\mathbf{q}]$ -algebra via the specialization $\mathbf{q} \mapsto q$) in such a way that $1 \otimes \tau$ is identified with f_1 . Since q can take infinitely many values, the identity in Proposition 2 follows.

5. Setting $f_t = 0$ for $t > n$, we see that the identity in Lemma 3 remains valid for any $t \geq 0$. We see that subspace of \mathcal{H} spanned by $\{f_t; t \geq 0\}$ coincides with the subspace spanned by $\{f_1^t; t \geq 0\}$; in particular it is a commutative subring.

6. Consider the endomorphism of $\mathbf{Q}(\mathbf{q}) \otimes_{\mathbf{Z}[\mathbf{q}]} H$ given by left multiplication by τ . Proposition 2 shows that the eigenvalues of this endomorphism are in

$$\{0, 1, 1 + \mathbf{q}, 1 + \mathbf{q} + \mathbf{q}^2, \dots, 1 + \mathbf{q} + \cdots + \mathbf{q}^{n-3}, 1 + \mathbf{q} + \cdots + \mathbf{q}^{n-1}\}.$$

The multiplicity of the eigenvalue $1 + \mathbf{q} + \cdots + \mathbf{q}^{k-1}$ is preserved by the specialization $\mathbf{q} = 1$ hence it is the same as the multiplicity of the eigenvalue k for the left multiplication by \mathbf{t} on $\mathbf{C}[S_n]$, which by [DFP] is the number of permutations of $[1, n]$ with exactly k fixed points.

REFERENCES

- [DFP] P.Diaconis, J.A.Fill and J.Pitman, *Analysis of top to random shuffles*, Combinatorics, probability and computing **1**(1992), 135-155.
- [GW] A.M.Garsia and N.Wallach, *Qsym over Sym is free* (2002).
- [Ph] R.M.Phatarfod, *On the matrix occuring in a linear search problem*, Jour.Appl.Prob. **28** (1991), 336-346.
- [W] N.Wallach, *Lie algebra cohomology and holomorphic continuation of generalized Jacquet integrals*, Advanced Studies in Pure Math. **14**(1988), 123-151.

DEPARTMENT OF MATHEMATICS, M.I.T., CAMBRIDGE, MA 02139